

ILL-POSEDNESS OF THE PRANDTL EQUATIONS IN SOBOLEV SPACES AROUND A SHEAR FLOW WITH GENERAL DECAY

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ABSTRACT. Motivated by the paper [6] [JAMS, 2010] about the linear ill-posedness for the Prandtl equations around a shear flow with exponential decay in normal variable, and the recent study of well-posedness on the Prandtl equations in Sobolev spaces, this paper aims to extend the result in [6] to the case when the shear flow has general decay. The key observation is to construct an approximate solution that captures the initial layer to the linearized problem motivated by the precise formulation of solutions to the inviscid Prandtl equations.

1. INTRODUCTION AND MAIN RESULTS

The Prandtl equations were introduced by Ludwig Prandtl [18] in 1904 to describe the motion of fluid with small viscosity near a solid boundary with non-slip boundary condition. This seminal work sets the foundation of boundary layer theories. Even though the Prandtl equations have been proved its importance in physics and engineering applications, the mathematical theories established are far from being satisfactory.

One of the pioneering works by Oleinik and her collaborators [17] in 1960s shows that under the monotonicity condition of the tangential velocity component in the normal direction to the boundary, local well-posedness theories of Prandtl equations can be established. This result was recently further improved in the framework of Sobolev spaces, cf. [1, 15]. On the other hand, the ill-posedness of this system in the Sobolev spaces for perturbation of a shear flow with a non-degenerate critical point was proved in the interesting paper [6] linearly and then nonlinear in [7, 9], following the long time study on the instability by many authors, cf. [5, 8, 14, 20] ect. It is noted that in the work [6], the shear flow is assumed to be exponentially decay to the uniform Euler flow in the normal direction with respect to the boundary. However, as pointed out in [11], the exponential decay should not be essential, in particular, in the physical consideration. Therefore, it remains the question whether the instability showed in [6] for exponential decay shear flow holds with general decay. In fact, the answer to this question in some sense reveals the monotonicity condition on the tangential velocity component is a necessary and sufficient condition for well-posedness in the framework of Sobolev spaces.

In the following, we will first present the result for the Prandtl equations in a two dimensional domain $\Omega \triangleq \{(t, x, y) : t > 0, (x, y) \in \mathbb{T} \times \mathbb{R}^+\}$, and then in the last section, we will give some discussion on the case in three space dimensions. That

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is, consider

$$(1.1) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u + \partial_x P - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ (u, v)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \end{cases} \quad \text{in } \Omega,$$

where $U = U(t, x)$ and $P = P(t, x)$ are the tangential velocity and pressure of the Euler flow adjacent to the boundary layer. Moreover, $U(t, x)$ and $P(t, x)$ satisfy the Bernoulli equation:

$$\partial_t U + U \partial_x U + \partial_x P = 0.$$

Since we are interested in the instability structure of this system around a shear flow, as in [6], we consider the simple case of (1.1) when the Euler flow U is constant:

$$U(t, x) \equiv U_0, \quad \text{and then,} \quad \partial_x P(t, x) \equiv 0.$$

In this case, the problem (1.1) becomes

$$(1.2) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ (u, v)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U_0. \end{cases} \quad \text{in } \Omega,$$

Note that (1.2) has a special shear flow solution $(u_s(t, y), 0)$, where the function $u_s(t, y)$ is a smooth solution to the following heat equation:

$$(1.3) \quad \begin{cases} \partial_t u_s - \partial_y^2 u_s = 0, & \text{in } \Omega, \\ u_s|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u_s = U_0, \\ u_s|_{t=0} = U_s(y) \end{cases}$$

with an initial shear layer $U_s(y)$. Then, we consider the linearization of the problem (1.2) around the shear flow $(u_s(t, y), 0)$, and obtain

$$(1.4) \quad \begin{cases} \partial_t u + u_s \partial_x u + v \partial_y u_s - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ (u, v)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 0. \end{cases} \quad \text{in } \Omega,$$

In [6], the authors showed that if the initial data $U_s(y)$ of the shear flow has a non-degenerate critical point, then the linear problem (1.4) is ill-posed in the case that $u_s - U_0$ exponentially decays to zero as $y \rightarrow +\infty$. The goal of this paper is to show that the exponential decay condition is not necessary. Indeed, a physical quantity that measures the effect of the boundary layer matching the outer flow, called displacement thickness, cf. [2, p.311], is defined by

$$(1.5) \quad \delta(t, x) = \int_0^\infty \left(1 - \frac{u(t, x, y)}{U(t, x)}\right) dy.$$

Hence, the finiteness of the displacement thickness only requires the integrability of the above function, which admits general decay of $u(t, x, y)$ to $U(t, x)$ when y tends to infinity.

To continue, let us first introduce some notations. Denote by $T(t, s)$ the linear solution operator:

$$(1.6) \quad T(t, s)u_0 := u(t, \cdot),$$

where u is the solution to the problem (1.4) with $u|_{t=s} = u_0$. Also, for any $\alpha, m \geq 0$, denote

$$\begin{aligned} W_\alpha^{m,\infty}(\mathbb{R}^+) &:= \{f = f(y), y \in \mathbb{R}^+; \|f\|_{W_\alpha^{m,\infty}} \triangleq \|e^{\alpha y} f(y)\|_{W^{m,\infty}(\mathbb{R}^+)} < \infty\}, \\ \mathcal{H}_\alpha^m &:= \{f = f(x, y), (x, y) \in \mathbb{T} \times \mathbb{R}^+; \|f\|_{\mathcal{H}_\alpha^m} \triangleq \|f(\cdot)\|_{H^m(\mathbb{T}_x, W_\alpha^{0,\infty}(\mathbb{R}_y^+))} < \infty\}. \end{aligned}$$

The main result on the linear ill-posedness of the Prandtl equations can be stated as follows.

Theorem 1. *Let $u_s(t, y)$ be the solution of the problems (1.3) satisfying*

$$u_s - U_0 \in C^0(\mathbb{R}^+; W_0^{4,\infty}(\mathbb{R}^+)) \cap C^1(\mathbb{R}^+; W_0^{2,\infty}(\mathbb{R}^+)),$$

and assume that the initial shear layer $U_s(y)$ has a non-degenerate critical point in \mathbb{R}^+ . Then, there exists $\sigma > 0$ such that for all $\delta > 0$,

$$(1.7) \quad \sup_{0 \leq s < t \leq \delta} \|e^{-\sigma(t-s)} \sqrt{|\partial_x|} T(t, s)\|_{\mathcal{L}(\mathcal{H}_\alpha^m, \mathcal{H}_0^{m-\mu})} = +\infty, \quad \forall \alpha, m \geq 0, \mu \in [0, \frac{1}{2}).$$

One consequence of the above theorem gives

Corollary 2. *Under the assumptions of Theorem 1, it holds that for any $\delta > 0$ and $\alpha, m \geq 0$,*

$$(1.8) \quad \sup_{0 \leq s < t \leq \delta} \|T(t, s)\|_{\mathcal{L}(\mathcal{H}_\alpha^m, \mathcal{H}_0^0)} = +\infty.$$

At the end of the introduction, let us mention that most of the mathematical theories for the Prandtl equations before 2000 can be found in the excellent review article [4]. In addition to those works mentioned before, some other interesting works can be found in [12, 13] for the three space dimensional Prandtl equations with special structure to avoid the secondary flow, cf. [16], the works in the framework of analytic function space in [3, 19, 22], and the existence of global weak solutions in [13, 21].

The rest of the paper will be arranged as follows. The main result on the linear instability for the system around a shear flow with general decay and a non-degenerate critical point will be proved in the next section by a new construction of approximate solutions. Some discussions on the case in three space dimensions will be given in the last section.

2. LINEAR INSTABILITY

In the following three subsections, we will prove Theorem 1 for the linear instability of the Prandtl equations.

2.1. Instability mechanism. In this subsection, we firstly recall the linear ill-posedness result in [6] about the linear instability mechanism of Prandtl equations, and then introduce the new approximate solutions for general decay shear flow. The key observation in [6] is to construct an unstable approximate solution to (1.4), in high frequency in the tangential variable x , with exponential growth in time t . To illustrate this kind of instability mechanism, as in [6], one can first replace the background shear flow in (1.4) by its initial data, and consider the following simpler

problem with coefficients independent of the variable t :

$$(2.1) \quad \begin{cases} \partial_t u + U_s \partial_x u + v U'_s - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ (u, v)|_{y=0} = 0, \end{cases} \quad \text{in } \Omega, \quad \lim_{y \rightarrow +\infty} u = 0.$$

Denote by \mathcal{L}_s the linearized Prandtl operator in (2.1) around the shear flow $(U_s(y), 0)$:

$$(2.2) \quad \mathcal{L}_s u := U_s \partial_x u + v U'_s - \partial_y^2 u, \quad \text{with } v(t, x, y) = - \int_0^y \partial_x u(t, x, z) dz.$$

In Section 2 of [6], the authors construct an approximate solution of (2.1), which has high x -frequency of the order ϵ^{-1} and grows in t exponentially at the rate of $\epsilon^{-\frac{1}{2}}$ for $\epsilon \ll 1$. Precisely, one can look for solutions to (2.1) in the form

$$(u, v)(t, x, y) = e^{i\epsilon^{-1}(x+w_\epsilon t)} (u_\epsilon(y), \epsilon^{-1}v_\epsilon(y)).$$

By plugging this into (2.1), the divergence free condition gives $u_\epsilon(y) = i v'_\epsilon(y)$, and then the first equation of (2.1) yields

$$(2.3) \quad \begin{cases} (w_\epsilon + U_s(y))v'_\epsilon(y) - U'_s(y)v_\epsilon(y) + i\epsilon v_\epsilon^{(3)}(y) = 0, & y > 0, \\ v_\epsilon|_{y=0} = v'_\epsilon|_{y=0} = 0. \end{cases}$$

Let $a > 0$ be a non-degenerate critical point of the initial shear layer $U_s(y)$, the following result was proved in [6].

Proposition 1. *There exists an approximate solution $(u_\epsilon^{app}, v_\epsilon^{app})(t, x, y)$ to the problem (2.1) in the form of*

$$(2.4) \quad (u_\epsilon^{app}, v_\epsilon^{app})(t, x, y) = e^{i\epsilon^{-1}(x+w_\epsilon t)} (i v'_\epsilon(y), \epsilon^{-1}v_\epsilon(y)),$$

where

$$(2.5) \quad w_\epsilon = -U_s(a) + \epsilon^{\frac{1}{2}}\tau$$

for some constant $\tau \in \mathbb{C}$ with the imaginary part $\Im \tau < 0$, and $v_\epsilon(y) \in W_0^{3,\infty}(\mathbb{R}^+)$, such that the error term $r_\epsilon^{app} := \partial_t u_\epsilon^{app} + \mathcal{L}_s u_\epsilon^{app}$ satisfies

$$(2.6) \quad r_\epsilon^{app}(t, x, y) = e^{i\epsilon^{-1}(x+w_\epsilon t)} R_\epsilon^{app}(y), \quad R_\epsilon^{app}(y) \in W_0^{0,\infty}(\mathbb{R}^+).$$

In fact, as shown in [6], the function $v_\epsilon(y)$ can be divided into a "regular" part $v_\epsilon^{reg}(y)$ and a "shear layer" part $v_\epsilon^{sl}(y)$, i.e.,

$$(2.7) \quad \begin{aligned} v_\epsilon(y) &= v_\epsilon^{reg}(y) + v_\epsilon^{sl}(y) \\ &= H(y-a) [U_s(y) - U_s(a) + \epsilon^{\frac{1}{2}}\tau] + \epsilon^{\frac{1}{2}} V\left(\frac{y-a}{\epsilon^{\frac{1}{4}}}\right). \end{aligned}$$

Here, H is the Heaviside function, and the shear layer profile $V(z)$ solves the following ODE:

$$(2.8) \quad \begin{cases} \left(\tau + U_s''(a)\frac{z^2}{2}\right)V' - U_s''(a)zV + iV^{(3)} = 0, & z \neq 0, \\ [V]|_{z=0} = -\tau, \quad [V']|_{z=0} = 0, \quad [V'']|_{z=0} = -U_s''(a), \\ \lim_{z \rightarrow \pm\infty} V = 0, & \text{exponentially,} \end{cases}$$

where the complex constant τ is the same as the one in (2.5), and the notation $[u]|_{z=0} := \lim_{\delta_1 \rightarrow 0+} u(\delta_1) - \lim_{\delta_2 \rightarrow 0-} u(\delta_2)$ denotes the jump of a related function $u(z)$

across $z = 0$. One can check that by virtue of w_ϵ given in (2.5), the function $v_\epsilon(y)$ defined in (2.7) solves the problem (2.3) except for the $O(\epsilon)$ -term coming from diffusion. Consequently, the corresponding approximate solution (2.4) admits the $O(\epsilon^{-1})$ -terms of the first equation of (2.1), which implies the estimate (2.6) automatically. Indeed, the direct calculation gives the expression of the error term $R_\epsilon^{app}(y)$ defined in (2.6):

$$(2.9) \quad \begin{aligned} R_\epsilon^{app}(y) = & -\epsilon^{-1} \left[U_s(y) - U_s(a) - U_s''(a) \frac{(y-a)^2}{2} \right] (v_\epsilon^{sl})'(y) \\ & + \epsilon^{-1} \left[U_s'(y) - U_s''(a)(y-a) \right] v_\epsilon^{sl}(y) - i(v_\epsilon^{reg})^{(3)}(y), \end{aligned}$$

so that the estimate of $R_\epsilon^{app}(y)$ in (2.6) follows from the exponential decay of the profile $V(z)$. Furthermore, we have from (2.9),

$$R_\epsilon^{app}(y) + i(v_\epsilon^{reg})^{(3)}(y) \in W_\alpha^{0,\infty}(\mathbb{R}^+), \text{ for any } \alpha \geq 0.$$

Note that the term $i(v_\epsilon^{reg})^{(3)}(y)$ does not appear in the error term when the background profile is the shear flow $u_s(t, y)$, not the initial shear layer $U_s(y)$, because of the heat equation, cf. (2.36).

In addition, we refer to [6] and note that the pair $(\tau, V(z))$ takes the following form:

$$(2.10) \quad \begin{cases} \tau &= \left| \frac{U_s''(a)}{2} \right|^{\frac{1}{2}} \tilde{\tau}, \\ V(z) &= \left| \frac{U_s''(a)}{2} \right|^{\frac{1}{2}} \left[\left(\tilde{\tau} + \left| \frac{U_s''(a)}{2} \right|^{\frac{1}{2}} z^2 \right) W \left(\left| \frac{U_s''(a)}{2} \right|^{\frac{1}{4}} z \right) - 1_{\mathbb{R}^+} \left(\tilde{\tau} + \left| \frac{U_s''(a)}{2} \right|^{\frac{1}{2}} z^2 \right) \right], \end{cases}$$

where the function $W(z)$ is a smooth solution of the following third order ordinary differential equation:

$$(2.11) \quad \begin{cases} \left(\tilde{\tau} + \text{sign}(U_s''(a)) z^2 \right)^2 \frac{d}{dz} W + i \frac{d^3}{dz^3} \left(\left(\tilde{\tau} + \text{sign}(U_s''(a)) z^2 \right) W \right) = 0, \\ \lim_{z \rightarrow -\infty} W = 0, \quad \lim_{z \rightarrow +\infty} W = 1. \end{cases}$$

The approximate solution $(u_\epsilon^{app}, v_\epsilon^{app})(t, x, y)$ given in (2.4) can be used to prove the instability of the problem (2.1) because the expression (2.4) combining with the property of the parameter $\tau : \Im \tau < 0$ implies a growing mode $e^{-\frac{\Im \tau}{\sqrt{\epsilon}}}$ for the approximation $(u_\epsilon^{app}, v_\epsilon^{app})(t, x, y)$ when $\epsilon \ll 1$. However, plugging the formula (2.7) of $v_\epsilon(y)$ into (2.4) yields

$$(2.12) \quad u_\epsilon^{app}(t, x, y) = e^{i\epsilon^{-1}(x+w_\epsilon t)} v_\epsilon'(y), \quad v_\epsilon'(y) = H(y-a) U_s'(y) + \epsilon^{\frac{1}{4}} V' \left(\frac{y-a}{\epsilon^{\frac{1}{4}}} \right).$$

Then, it implies that the approximation $u_\epsilon^{app}(t, x, y)$ has the same decay rate as $U_s'(y)$ when $y \rightarrow +\infty$. In particular, $u_\epsilon^{app} \notin \mathcal{H}_\alpha^m$ initially for any $\alpha > 0$ if $U_s'(y)$ does not decay exponentially as $y \rightarrow +\infty$.

Therefore, to study the case of shear flow with general decay, the above approximation u_ϵ^{app} in (2.12) will be inappropriate since the operator we consider now is

$$T(t, s) : \mathcal{H}_\alpha^{m_1} \mapsto \mathcal{H}_0^{m_2}, \quad \forall \alpha > 0, \text{ for some } m_1, m_2 > 0.$$

For this, we need to modify the construction of approximate solution (2.4) with (2.7) to problem (2.1), in order that at least the initial tangential data of the

approximation has an exponential decay rate as $y \rightarrow +\infty$. So, we will look for a new approximate solution of (2.1) in the following form:

$$(2.13) \quad (\tilde{u}_\epsilon^{app}, \tilde{v}_\epsilon^{app})(t, x, y) = e^{i\epsilon^{-1}(x+\bar{w}_\epsilon t)} \left(i v'_{\epsilon,1}(y) + i t v'_{\epsilon,2}(y), \epsilon^{-1} v_{\epsilon,1}(y) + \epsilon^{-1} t v_{\epsilon,2}(y) \right).$$

In the above expression, we expect that, on one hand,

$$(2.14) \quad \tilde{w}_\epsilon = w_\epsilon, \quad v_{\epsilon,2}(y) = v_\epsilon(y),$$

where w_ϵ and $v_\epsilon(y)$ are given in Proposition 1, thus the instability of (2.1) preserves through the eigenvalue perturbation τ as mentioned above; on the other hand,

$$(2.15) \quad v_{\epsilon,1}(0) = v'_{\epsilon,1}(0) = 0, \quad \lim_{y \rightarrow +\infty} v'_{\epsilon,1}(y) = 0, \text{ exponentially,}$$

so that the initial data of $\tilde{u}_\epsilon^{app}(t, x, y)$ given by (2.13) has an exponential decay rate as $y \rightarrow +\infty$.

The motivation of the construction in the form of (2.13) comes from the expression of solutions to the linearized inviscid Prandtl equation around a shear flow $(U(y), 0)$. That is, the system

$$(2.16) \quad \begin{cases} \partial_t u + U(y) \partial_x u + U'(y) v = 0, \\ \partial_x u + \partial_y v = 0, \\ v|_{y=0} = 0, \quad u|_{t=0} = u_0(x, y) \end{cases}$$

has the solution

$$(2.17) \quad \begin{aligned} u(t, x, y) &= u_0(x - tU(y), y) + tU'(y) \int_0^y u_{0x}(x - tU(z), z) dz, \\ v(t, x, y) &= - \int_0^y \left\{ u_{0x}(x - tU(z), z) + t[U(y) - U(z)] u_{0xx}(x - tU(z), z) \right\} dz, \end{aligned}$$

see Proposition 5.1 in [10]. From the above expression (2.17), we know that when $t > 0$, the decay rate of tangential velocity of the solution to the problem (2.16) is not faster than the one of background shear flow $U'(y)$, even though the initial data $u_0(x, y)$ can decay very rapidly as $y \rightarrow +\infty$.

Now, it remains to find a suitable $v_{\epsilon,1}(y)$ for the new approximation (2.13), such that the error term

$$\tilde{r}_\epsilon^{app} := \partial_t \tilde{u}_\epsilon^{app} + \mathcal{L}_s \tilde{u}_\epsilon^{app}$$

still satisfies the relation (2.6). By virtue of (2.14), a direct computation yields that $\tilde{r}_\epsilon^{app}(t, x, y) = e^{i\epsilon^{-1}(x+w_\epsilon t)} \tilde{R}_\epsilon^{app}(y)$ and

$$(2.18) \quad \begin{aligned} \tilde{R}_\epsilon^{app}(y) &= -\epsilon^{-1}[w_\epsilon + U_s(y)]v'_{\epsilon,1}(y) + \epsilon^{-1}U'_s(y)v_{\epsilon,1}(y) + i v_{\epsilon,1}^{(3)}(y) + i v'_\epsilon(y) + t R_\epsilon^{app}(y) \\ &:= \bar{R}_\epsilon^{app}(y) + t R_\epsilon^{app}(y) \end{aligned}$$

with $R_\epsilon^{app}(y)$ given by (2.9). Note that

$$-\epsilon^{-1}[w_\epsilon + U_s(y)]v'_{\epsilon,1}(y) + i v_{\epsilon,1}^{(3)}(y) \in W_\alpha^{0,\infty}(\mathbb{R}^+)$$

provided that $v'_{\epsilon,1}(y) \in W_\alpha^{0,\infty}(\mathbb{R}^+)$ for some $\alpha > 0$. Thus, to ensure $\bar{R}_\epsilon^{app}(y) \in W_\alpha^{0,\infty}(\mathbb{R}^+)$, we only need

$$\epsilon^{-1}U'_s(y)v_{\epsilon,1}(y) + i v'_\epsilon(y) \in W_\alpha^{0,\infty}(\mathbb{R}^+),$$

which implies that by combining with (2.12),

$$(2.19) \quad v_{\epsilon,1}(y) \rightarrow -i\epsilon \quad \text{exponentially, as } y \rightarrow +\infty.$$

Obviously, for any function $f(y)$, $y \in \mathbb{R}^+$:

$$(2.20) \quad f(y) \in C_c^\infty(\mathbb{R}^+), \quad \int_0^{+\infty} f(y)dy \neq 0,$$

the function

$$(2.21) \quad v_{\epsilon,1}(y) := -i\epsilon \frac{\int_0^y f(z)dz}{\int_0^{+\infty} f(y)dy}$$

meets the requirements (2.15) and (2.19). Then, plugging the above expression (2.21) into (2.13), we obtain the new approximate solution to (2.1):

$$(2.22) \quad (\tilde{u}_\epsilon^{app}, \tilde{v}_\epsilon^{app})(t, x, y) = e^{i\epsilon^{-1}(x+w_\epsilon t)} \left(\frac{\epsilon f(y)}{\int_0^{+\infty} f(y)dy} + itv'_\epsilon(y), -\frac{i \int_0^y f(z)dz}{\int_0^{+\infty} f(y)dy} + \epsilon^{-1}tv_{\epsilon,2}(y) \right),$$

where the functions $v_\epsilon(y)$ and $f(y)$ are given by (2.7) and (2.20) respectively.

2.2. Construction of approximate solutions. Following the construction of approximate solutions to the simplified problem (2.1) given in the previous subsection, and also by the arguments used in [6], we are going to construct the approximate solutions to the original linearized problem (1.4). Since the approximate solutions to (2.1) given in (2.22) are obtained with the background state being frozen at the initial data $u_s|_{t=0} = U_s(y)$, to construct the approximate solutions of the original problem (1.4) with background state being shear flow in the time interval $0 < t < t_0$, we need some modification as in [6].

Let $u_s(t, y)$ satisfy the assumptions of Theorem 1, and $a > 0$ be a non-degenerate critical point of $U_s(y)$. Without loss of generality, we assume that $U_s''(a) < 0$, then the differential equation

$$(2.23) \quad \begin{cases} \partial_t \partial_y u_s(t, a(t)) + \partial_y^2 u_s(t, a(t)) a'(t) = 0, \\ a(0) = a \end{cases}$$

defines a non-degenerate critical point $a(t)$ of $u_s(t, \cdot)$ when $0 < t < t_0$ for some small $t_0 > 0$. Moreover, we have $\partial_y^2 u_s(t, a(t)) < 0$ for all $t \in [0, t_0)$ with t_0 small enough. As in [6], we take $\tau, W(z)$ given by (2.11) (we drop the tilde of $\tilde{\tau}$ for brevity), and set

$$(2.24) \quad V(z) := (\tau - z^2)W(z) - 1_{\mathbb{R}^+}(\tau - z^2).$$

For $0 < \epsilon \ll 1$, introduce

$$(2.25) \quad w_\epsilon(t) := -u_s(t, a(t)) + \epsilon^{\frac{1}{2}} \left| \frac{\partial_y^2 u_s(t, a(t))}{2} \right|^{\frac{1}{2}} \tau,$$

and the “regular” part of the tangential velocity field

$$(2.26) \quad v_\epsilon^{reg}(t, y) = H(y - a(t)) \left[u_s(t, y) - u_s(t, a(t)) + \epsilon^{\frac{1}{2}} \left| \frac{\partial_y^2 u_s(t, a(t))}{2} \right|^{\frac{1}{2}} \tau \right],$$

as well as the “shear layer” part

(2.27)

$$v_\epsilon^{sl}(t, y) := \epsilon^{\frac{1}{2}} \varphi(y - a(t)) \left| \frac{\partial_y^2 u_s(t, a(t))}{2} \right|^{\frac{1}{2}} V \left(\left| \frac{\partial_y^2 u_s(t, a(t))}{2} \right|^{\frac{1}{4}} \cdot \frac{y - a(t)}{\epsilon^{\frac{1}{4}}} \right).$$

Here, φ is a smooth truncation function near 0, and $V(z)$ is given in (2.24). Also, for any function $f(y)$ satisfying (2.20), let

$$(2.28) \quad \tilde{v}_\epsilon(y) := \frac{\int_0^y f(z) dz}{\int_0^{+\infty} f(y) dy}.$$

Next, according to the discussion in the above subsection, the approximate solution of the problem (1.4) can be defined as follows:

$$(2.29) \quad (u_\epsilon, v_\epsilon)(t, x, y) = e^{i\epsilon^{-1}x} (U_\epsilon, V_\epsilon)(t, y)$$

with

$$(2.30) \quad \begin{aligned} U_\epsilon(t, y) &= e^{i\epsilon^{-1} \int_0^t w_\epsilon(s) ds} \left[\epsilon \tilde{v}'_\epsilon(y) + it \partial_y (v_\epsilon^{reg}(t, y) + v_\epsilon^{sl}(t, y)) \right], \\ V_\epsilon(t, y) &= e^{i\epsilon^{-1} \int_0^t w_\epsilon(s) ds} \left[-i \tilde{v}_\epsilon(y) + \epsilon^{-1} t (v_\epsilon^{reg}(t, y) + v_\epsilon^{sl}(t, y)) \right]. \end{aligned}$$

For the function $(u_\epsilon, v_\epsilon)(t, x, y)$ in (2.29) to be 2π -periodic in x , we take $\epsilon = \frac{1}{n}$ with $n \in \mathbb{N}$. It is straightforward to check that,

$$(u_\epsilon, v_\epsilon)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u_\epsilon = 0,$$

and the divergence free condition holds. Also, $u_\epsilon(t, x, y) = e^{i\epsilon^{-1}x} U_\epsilon(t, y)$ is analytic in the tangential variable x and $W^{2,\infty}$ in y . Moreover, there are positive constants C_0 and σ_0 , independent of ϵ , such that

$$(2.31) \quad \|U_\epsilon(t, \cdot)\|_{W_0^{2,\infty}} \leq C_0 e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}, \quad t \in [0, t_0],$$

in particular,

$$(2.32) \quad \|U_\epsilon(0, \cdot)\|_{W_\alpha^{2,\infty}} \leq C_0 \epsilon, \quad \forall \alpha \geq 0.$$

Plugging the relation (2.29) into the original linearized Prandtl equations (1.4), it follows that

$$(2.33) \quad \begin{cases} \partial_t u_\epsilon + u_s \partial_x u_\epsilon + v_\epsilon \partial_y u_s - \partial_y^2 u_\epsilon = r_\epsilon, \\ \partial_x u_\epsilon + \partial_y v_\epsilon = 0, \\ (u_\epsilon, v_\epsilon)|_{y=0} = 0. \end{cases} \quad \text{in } \Omega,$$

The remainder term r_ϵ can be represented by $r_\epsilon(t, x, y) = e^{i\epsilon^{-1}x} R_\epsilon(t, y)$ and

$$(2.34) \quad R_\epsilon(t, y) := \bar{R}_\epsilon(t, y) + t \tilde{R}_\epsilon(t, y),$$

where

$$(2.35) \quad \begin{aligned} \bar{R}_\epsilon(t, y) &= e^{i\epsilon^{-1} \int_0^t w_\epsilon(s) ds} \left\{ i [w_\epsilon(t) + u_s(t, y)] \tilde{v}'_\epsilon(y) - i \partial_y u_s(t, y) \tilde{v}_\epsilon(y) - \epsilon \tilde{v}_\epsilon^{(3)}(y) \right. \\ &\quad \left. + i \partial_y (v_\epsilon^{reg}(t, y) + v_\epsilon^{sl}(t, y)) \right\}, \end{aligned}$$

and

(2.36)

$$\begin{aligned} \tilde{R}_\epsilon(t, y) = e^{i\epsilon^{-1} \int_0^t w_\epsilon(s) ds} \Big\{ & -\epsilon^{-1} \left[u_s(t, y) - u_s(t, a(t)) - \partial_y^2 u_s(t, a(t)) \frac{(y - a(t))^2}{2} \right] \partial_y v_\epsilon^{sl}(t, y) \\ & + \epsilon^{-1} \left[\partial_y u_s(t, y) - \partial_y^2 u_s(t, a(t))(y - a(t)) \right] v_\epsilon^{sl}(t, y) \\ & + i\partial_t \partial_y v_\epsilon^{sl}(t, y) + O(\epsilon^\infty) \Big\}. \end{aligned}$$

The term $O(\epsilon^\infty)$ in (2.36) represents the part of remainder with exponential decay in y that comes from the fact that $V(z)$ decays exponentially and the derivatives of $\varphi(\cdot - a(t))$ vanish outside a neighborhood of $a(t)$. Combining the formulation (2.36) of $\tilde{R}_\epsilon(t, y)$ and the exponential decay of $v_\epsilon^{sl}(t, y)$ yields

$$(2.37) \quad \|\tilde{R}_\epsilon(t, \cdot)\|_{W_\alpha^{0,\infty}} \leq C_1 e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}, \quad \forall \alpha \geq 0$$

with the constant $\sigma_0 > 0$ given in (2.31). On the other hand, from (2.26)-(2.28) we have

$$-i\partial_y u_s(t, y) \tilde{v}_\epsilon(y) + i\partial_y (v_\epsilon^{reg}(t, y) + v_\epsilon^{sl}(t, y)) \equiv 0, \quad \text{for large } y > 0,$$

and then,

$$-i\partial_y u_s(t, y) \tilde{v}_\epsilon(y) + i\partial_y (v_\epsilon^{reg}(t, y) + v_\epsilon^{sl}(t, y)) \in W_\alpha^{2,\infty}, \quad \forall \alpha \geq 0,$$

which implies that the estimate (2.37) also holds for the term $\tilde{R}_\epsilon(t, y)$. Thus, with the same σ_0 given in (2.31), the term $R_\epsilon(t, y)$ satisfies

$$(2.38) \quad \|R_\epsilon(t, \cdot)\|_{W_\alpha^{0,\infty}} \leq C_1 e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}, \quad \forall \alpha \geq 0,$$

where the constant $C_1 > 0$ is independent of ϵ .

2.3. Proof of the main Theorem. Based on the approximate solutions constructed in the above subsection, we can apply the approach in [6] to prove Theorem 1. We now sketch the proof as follows.

The proof is based on the verification of (1.7) for the tangential differential operator by contradiction. Suppose that (1.7) does not hold, that is, for all $\sigma > 0$, there exists $\delta > 0$, $\alpha_0, m \geq 0$ and $\mu \in [0, \frac{1}{2})$ such that

$$(2.39) \quad \sup_{0 \leq s < t \leq \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_x|}} T(t, x)\|_{\mathcal{L}(\mathcal{H}_{\alpha_0}^m, \mathcal{H}_0^{m-\mu})} < +\infty.$$

Introduce the operator

$$T_\epsilon(t, s) : W_{\alpha_0}^{0,\infty}(\mathbb{R}^+) \mapsto W_0^{0,\infty}(\mathbb{R}^+)$$

as

$$(2.40) \quad T_\epsilon(t, s) U_0 := e^{-i\epsilon^{-1}x} T(t, s) \left(e^{i\epsilon^{-1}x} U_0 \right)$$

with $T(t, s)$ being defined in (3.4). From (2.39), we have

$$(2.41) \quad \|T_\epsilon(t, s)\|_{\mathcal{L}(W_{\alpha_0}^{0,\infty}, W_0^{0,\infty})} \leq C_2 \epsilon^{-\mu} e^{\frac{\sigma(t-s)}{\sqrt{\epsilon}}}, \quad \forall 0 \leq s < t \leq \delta$$

for some constant $C_2 > 0$ independent of ϵ .

Next, denote by

$$L_\epsilon := e^{-i\epsilon^{-1}x} L e^{i\epsilon^{-1}x},$$

where L is the linearized Prandtl operator around the shear flow $(u^s(t, y), 0)$. Let $U(t, y)$ be a solution to the problem

$$\partial_t U + L_\epsilon U = 0, \quad U|_{t=0} = U_\epsilon(0, y),$$

where $U_\epsilon(t, y)$ is given in (2.30). Thus, we have

$$U(t, y) = T_\epsilon(t, 0)U_\epsilon(0, y),$$

and by using (2.32) and (2.41) it follows that

$$(2.42) \quad \|U(t, \cdot)\|_{W_0^{0,\infty}} \leq C_2 \epsilon^{-\mu} e^{\frac{\sigma t}{\sqrt{\epsilon}}} \|U_\epsilon(0, \cdot)\|_{W_{\alpha_0}^{0,\infty}} \leq C_3 \epsilon^{1-\mu} e^{\frac{\sigma t}{\sqrt{\epsilon}}}, \quad \forall t \in (0, \delta]$$

for some constant $C_3 > 0$ independent of ϵ .

On the other hand, we know that the difference $\tilde{U} := U - U_\epsilon$ can be obtained by the Duhamel principle:

$$(2.43) \quad \tilde{U}(t, \cdot) = \int_0^t T_\epsilon(t, s) R_\epsilon(s, \cdot) ds, \quad \forall t \leq \delta.$$

From (2.38), (2.41) and (2.43), and choosing $\sigma < \sigma_0$, we have

$$(2.44) \quad \|\tilde{U}(t, \cdot)\|_{W_0^{0,\infty}} \leq C_1 C_2 \epsilon^{-\mu} \int_0^t e^{\frac{\sigma(t-s)}{\sqrt{\epsilon}}} e^{\frac{\sigma_0 s}{\sqrt{\epsilon}}} ds \leq C_4 \epsilon^{\frac{1}{2}-\mu} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}},$$

where the constant $C_4 > 0$ is independent of ϵ . Then, by combining (2.44) with the expression of $U_\epsilon(t, y)$ in (2.30), we obtain that for $t \in (0, \delta]$ and sufficiently small ϵ ,

$$(2.45) \quad \begin{aligned} \|U(t, \cdot)\|_{W_0^{0,\infty}} &\geq \|U_\epsilon(t, \cdot)\|_{W_0^{0,\infty}} - \|\tilde{U}(t, \cdot)\|_{W_0^{0,\infty}} \\ &\geq e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}} (C_5 t - C_6 \epsilon) - C_4 \epsilon^{\frac{1}{2}-\mu} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}} \\ &\geq C_5 t e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}} - 2C_4 \epsilon^{\frac{1}{2}-\mu} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}. \end{aligned}$$

As $\sigma < \sigma_0$, comparing (2.42) with (2.45), the contradiction arises when $t \gg \frac{\mu |\ln \epsilon|}{\sigma_0 - \sigma} \epsilon^{\frac{1}{2}-\mu}$ with sufficiently small ϵ . Thus, the proof of Theorem 1 is completed.

3. FURTHER DISCUSSIONS

In this section, we point out that the above results can be extended to the three space dimensions under some condition on the background shear flow given in [14, Theorem 2.3]. More precisely, consider the three dimensional Prandtl equations in the domain $\{(t, x, y, z) : t > 0, (x, y) \in \mathbb{T}^2, z \in \mathbb{R}^+\}$:

$$(3.1) \quad \begin{cases} \partial_t u + (u \partial_x + v \partial_y + w \partial_z) u - \partial_z^2 u = 0, \\ \partial_t v + (u \partial_x + v \partial_y + w \partial_z) v - \partial_z^2 v = 0, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ (u, v, w)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u, v) = (U_0, V_0) \end{cases}$$

with positive constants U_0 and V_0 . Let $(u_s, v_s)(t, z)$ be a smooth solution of the heat equations:

$$(3.2) \quad \begin{cases} \partial_t u_s - \partial_z^2 u_s = 0, & \partial_t v_s - \partial_z^2 v_s = 0, \\ (u_s, v_s)|_{z=0} = 0, & \lim_{z \rightarrow +\infty} (u_s, v_s) = (U_0, V_0). \end{cases}$$

It is straightforward to verify that the shear velocity profile $(u_s, v_s, 0)(t, z)$ solves the problem (3.1). Then, we study the linearized problem of (3.1) around the shear flow $(u_s, v_s, 0)(t, z)$:

$$(3.3) \quad \begin{cases} \partial_t u + (u_s \partial_x + v_s \partial_y)u + w \partial_z u_s - \partial_z^2 u = 0, \\ \partial_t v + (u_s \partial_x + v_s \partial_y)v + w \partial_z v_s - \partial_z^2 v = 0, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ (u, v, w)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u, v) = 0. \end{cases}$$

Denote by $T(t, s)$ the linearized solution operator of problem (3.3), i.e.,

$$(3.4) \quad T(t, s)((u_0, v_0)) := (u, v)(t, \cdot),$$

where (u, v) is the solution of (3.3) with $(u, v)|_{t=s} = (u_0, v_0)$. The result on the linear instability of the three-dimensional Prandtl equations is:

Proposition 2. *Let $(u_s, v_s)(t, z)$ solve (3.2) with*

$$(u_s - U_0, v_s - V_0) \in C^0(\mathbb{R}^+; W_0^{4,\infty}(\mathbb{R}_z^+)) \cap C^1(\mathbb{R}^+; W_0^{2,\infty}(\mathbb{R}_z^+)),$$

and assume that the initial data $(U_s, V_s)(z) \triangleq (u_s, v_s)(0, z)$ satisfies that

$$(3.5) \quad \exists z_0 > 0, \text{ s.t. } V_s'(z_0)U_s''(z_0) \neq U_s'(z_0)V_s''(z_0).$$

Then, there exists $\sigma > 0$ such that for any $\delta > 0$,

$$(3.6) \quad \sup_{0 \leq s < t \leq \delta} \|e^{-\sigma(t-s)} \sqrt{|\partial_{\mathcal{T}}|} T(t, s)\|_{\mathcal{L}(H_\alpha^m, H_0^{m-\mu})} = +\infty, \quad \forall m, \alpha \geq 0, \mu \in [0, \frac{1}{4}),$$

where the operator $\partial_{\mathcal{T}}$ represents the tangential derivative ∂_x or ∂_y , and the weighted Sobolev spaces H_α^m are given by

$$H_\alpha^m := H^m(\mathbb{T}_{x,y}^2; W_\alpha^{0,\infty}(\mathbb{R}_z^+)), \quad \forall m, \alpha \geq 0.$$

Moreover,

$$(3.7) \quad \sup_{0 \leq s < t \leq \delta} \|T(t, s)\|_{\mathcal{L}(H_\alpha^m, H_0^0)} = +\infty, \quad \forall m, \alpha \geq 0.$$

This proposition can be proved by combining the above arguments with the analysis in [14], hence, we omit it for brevity.

Finally, the nonlinear instability in both 2D and 3D cases can also be discussed for the case when the background shear flow has general decay by using the above linear instability results and the arguments from [7, 9] and [14].

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